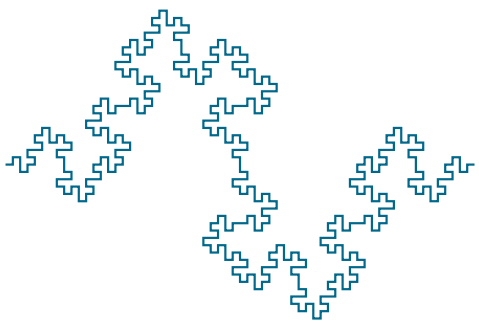
Cryptographic Mathematics I
Week 1

Dr. Eberhard Mayerhofer

University of Limerick



Semester I 2016/7

Structure

- 25% Project work, week 6-8.
- 5% attendance
- 70% exam
- contribution to tutorials is compulsory, if rejected there will be a cap of CA and attendance at 10%
- Syllabus: Book of Modules (www.bookofmodules.ul.ie)
- Lecture Notes and Exercise Sheets: Sulis (www.sulis.ul.ie)
- Office Hours: After Lab.
- Discuss timetable (www.timetable.ul.ie)

Definition

The Babylonians discovered **Pythagorean triples** i.e. numbers a, b, c satisfying

$$a^2 + b^2 = c^2.$$

- Most popular: $(3, 4, 5)$
- Others: $(5, 12, 13)$, $(7, 24, 25)$, $(8, 15, 17)$
- There are infinitely many solutions.

Introducing the rational numbers $x = a/c$ and $y = b/c$, we can write the equation as

$$x^2 + y^2 = 1.$$

Find all Pythagorean triples $\Leftrightarrow$ find all points on the unit circle (why?) which have rational coordinates (x, y)

There are four obvious points on the circle $x^2 + y^2 = 1$ (which?).

Geometric Approach

Let $P = (-1, 0)$ and let $m = p/q$, where p and q are integers and $q \neq 0$).
Consider the line through P with slope m . The equation of this line is

$$y - 0 = m(x + 1). \qquad (Why?)$$

This line will intersect the circle at a second point Q.
($\rightarrow$ see wonderful painting on whiteboard)

To find the coordinates of Q, we substitute $y = m(x+1)$ in the equation $x^2 + y^2 = 1$.

$$\begin{aligned} x^2 + m^2(x+1)^2 &= 1 \\ (x^2-1) + m^2(x+1)^2 &= 0 \\ (x-1)(x+1) + m^2(x+1)^2 &= 0 \\ (x+1)[x-1+m^2(x+1) &= 0 \end{aligned}$$

This means: $x = -1$ is a solution (clear, why?), or

$$x = \frac{1-m^2}{1+m^2}.$$

This is the x coordinate of Q.
Using this we obtain the y-coordinate of Q

$$y = m(x+1) = m\left(\frac{1-m^2}{1+m^2} + 1\right) = \frac{2m}{1+m^2}.$$

Thus

$$Q = \left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right).$$

Are these all rational solutions of $x^2 + y^2 = 1$?

- Since m is rational, the coordinates of Q are also rational.
- If we start with a point Q on the circle with rational coordinates then the slope of the line PQ will be rational.

Hence, we have determined all the points on the circle $x^2 + y^2 = 1$ with rational coordinates.

How to get all solutions of $a^2 + b^2 = c^2$?

Set $m = v/u$ with u, v integers. This leads us to the equation

$$(u^2 - v^2)^2 + (2uv)^2 = (u^2 + v^2)^2$$

from which we see that

$$a = u^2 - v^2 \qquad b = 2uv \qquad c = u^2 + v^2$$

is a Pythagorean triple. By substituting for u and v we can find Pythagorean triples, as shown in the table below.

u	2	3	4	4	5	5	6	6	7	7	7	8	8	8
v	1	2	1	3	2	4	1	5	2	4	6	1	3	5
a	3	5	15	7	21	9	35	11	45	33	13	63	55	49
b	4	12	8	24	20	40	12	60	28	56	84	16	48	80
c	5	13	17	25	29	41	37	61	53	65	85	65	73	89

- The triples (a, b, c) in table are such that a, b, c don't have a common factor.
- Of course, if (a, b, c) is a Pythagorean triple and k a positive integer, then (ka, kb, kc) is another Pythagorean triple.
- Fermat's Last Theorem:

$$a^n + b^n = c^n$$

has no solutions in positive integers if $n \geq 3$. Fermat conjectured this in the 17th century. Proved in 1994 by Andrew Wiles and Richard Taylor.

- Uses elliptic curves. Are important in cryptography (see later).

Definition

Let m and n be integers with $m \neq 0$. We say m divides n if n is a multiple of m i.e. there exists an integer k such that $n = km$.
If m divides n we write $m \mid n$.

Example

Thus

$$2 \mid 60 \text{ and } 3 \mid 60 \text{ and } 5 \mid 60 \text{ and } 10 \mid 60$$

A number that divides n is called a **divisor** of n.

Basic Observations

- All positive divisors of n are less than or equal to $|n|$.
- Hence only finite number of divisors.
- Always 1 is a divisor of every integer n.

Thus if we take two integers m and n, the set of integers which are divisors of both m and n is a non-empty finite set and has a largest element which is called the **greatest common divisor** of m and n abbreviated to $\gcd(m, n)$. If $\gcd(m, n) = 1$ we say m and n are **coprime**.

How do we calculate $\gcd(m, n)$?

- (basic, not efficient) list all the divisors of m and n and take the largest number which is in both lists.
- (not efficient, either): List all prime divisors, and find the common ones (including powers of the same prime)
- (**Euclidean algorithm**, efficient). It involves doing a sequence of divisions with remainder until the remainder is zero.

Division with remainder

- Let m and q be integers.
- We can always write $m = q \times n + r$.
  - m is the dividend
  - n is the divisor
  - q is the (integer!) quotient
  - r is the integer remainder
- Practice: we take the integer part of m/n as q and then calculate $r = m - n \times q$.

Euclidean Algorithm (Example 1)

Let $m = 126$ and $n = 1812$. Divide 126 into 1812.

$$\begin{aligned} 1812 &= 14 \times 126 + 48 \\ 126 &= 2 \times 48 + 30 \\ 48 &= 1 \times 30 + 18 \\ 30 &= 1 \times 18 + 12 \\ 18 &= 1 \times 12 + 6 \\ 12 &= 2 \times 6 \end{aligned}$$

The Euclidean algorithm tells you that when you find a remainder of 0 then the greatest common divisor is the remainder in the previous step. Hence

$$\gcd(1812, 126) = 6.$$

Why does the Euclidean algorithm work?

Assume $a > b > 0$ and let the sequence of steps be as follows:

$$\begin{aligned} a &= q_1 \times b + r_1 \\ b &= q_2 \times r_1 + r_2 \\ r_1 &= q_3 \times r_2 + r_3 \\ &\vdots \\ r_{n-2} &= q_n \times r_{n-1} + r_n \\ r_{n-1} &= q_{n+1} \times r_n + 0 \end{aligned}$$

For some n, $r_{n+1} = 0$ (Why?). Now $r_n \mid r_{n-1}$ and working upwards we find $r_n \mid r_{n-2}$... Eventually, we find that $r_n \mid r_1$ and $r_n \mid b$.

Finally from the first equation we find that $r_n \mid a$ and $r_n \mid b$ i.e. r_n is a common divisor of a and b.

And working down we see that any common divisor of a and b is a divisor of successively $r_1, r_2, r_3, \ldots, r_n$. Thus r_n is the greatest common divisor of a and b.

Efficiency

- Number of steps in the Euclidean algorithm $\leq$ seven times the number of digits in b.
- Thus Euclidean algorithm calculations are easily done on a computer with numbers having thousands of digits.
- Improved version of the Euclidean algorithm called the **binary GCD algorithm**. It is based on the fact that if m and n are both even then $\gcd(m,n) = 2\gcd(m/2, n/2)$ while if m is even and n odd, $\gcd(m,n) = \gcd(m/2, n)$. In binary, division by 2 is a simple shift operation.

Binary GCD algorithm ($\rightarrow$ Example 1)

$$
\begin{aligned}
\gcd(87654321, 12345678) &= \gcd(87654321, 6172839) \\
&= \gcd(14 \times 6172839 + 1234575, 6172839) \\
&= \gcd(6172839, 1234575) \\
&= \gcd(4 \times 1234575 + 1234539, 1234575) \\
&= \gcd(1234575, 1234539) \\
&= \gcd(1235439 + 36, 1234539) \\
&= \gcd(1234539, 36) \\
&= \gcd(1234539, 18) \\
&= \gcd(1234539, 9) \\
&= \gcd(9 \times 137171, 9) = 9
\end{aligned}
$$

Properties

1. If $m \mid a$ and $m \mid b$ then $m \mid a + b$ and $m \mid a - b$.
2. If $m \mid a$ then $m \mid ka$ where k is an integer. Combining with 1) we have if $m \mid a$ and $m \mid b$ then $m \mid ka \pm lb$ for any integers k and l.
3. If $a \mid b$ and $b \mid c$ then $a \mid c$.
4. If $a \mid b$ and $c \mid d$ then $ac \mid bd$.
5. If $m \neq 0$ then $a \mid b \iff ma \mid mb$.
6. If $d \mid a$ and $a \neq 0$ then $|d| \leq |a|$.
7. $a \mid b$ and $b \mid a$ if and only if $a = \pm b$

Let a and b be integers. One can ask the question: What integers can be written in the form $ax + by$ with x and y integers? Take $a = 24$ and $b = 66$. We can draw up a table of values of $24x + 66y$ for small values of x and y

$y\backslash x$	-4	-3	-2	-1	0	1	2	3	4
-4	-360	-336	-312	-288	-264	-240	-216	-192	-168
-3	-294	-270	-246	-222	-198	-174	-150	-126	-102
-2	-228	-204	-180	-156	-132	-108	-84	-60	-36
-1	-162	-138	-114	-90	-66	-42	-18	6	30
0	-96	-72	-48	-24	0	24	48	72	96
1	-30	-6	18	42	66	90	114	138	162
2	36	60	84	108	132	156	180	204	228
3	102	126	150	174	198	222	246	270	294
4	168	192	216	240	264	288	312	336	360

Observations

- Every entry in the table is a multiple of 6 which is $\gcd(24, 66)$
- 6 also appears in the table as

$$6 = 3 \times 24 - 1 \times 66.$$

- Conclude: smallest positive value in our table is the $\gcd(a, b)$.

This is true in general. Such an expression can be obtained from the Euclidean algorithm calculations to find the GCD.

$$66 = 2 \times 24 + 18 \qquad 66 - 2 \times 24 = 18$$
$$24 = 1 \times 18 + 6 \qquad 24 - 1 \times 18 = 6$$
$$18 = 3 \times 6 \qquad 18 - 3 \times 6 = 0$$

On the right we have rewritten the equations to have the remainders alone on the right hand side. Substituting in the second equation for 18 the left hand side of the first equation, we get

$$6 = 24 - 1 \times 18 = 24 - 1 \times (66 - 2 \times 24) = 3 \times 24 - 1 \times 66$$

One can also work forwards. From the first equation $18 = a - 2b$ where $a = 66$ and $b = 24$. The second equation becomes $b - (a - 2b) = 6$ from which we again obtain $6 = 3b - a = 3 \times 24 - 1 \times 66$.

- Thus the linear equation $ax + by = \gcd(a, b)$ has a solution in integers x and y.
- (special case $\gcd(a, b) = 1$). We can create other solutions as follows.

  $$ax_1+by_1 = 1 \Rightarrow ax_1+kab+by_1-kab = a(x_1+kb)+b(y_1-ka) = 1$$

  Thus $(x_1 + kb, y_1 - ka)$ is another solution.
- $(ax + by = d = \gcd(a, b))$. As d is a divisor of both a and b we can divide both sides of our equation by d to give

  $$\frac{a}{d}x + \frac{b}{d}y = 1$$

  and apply what we have just proved.

As result we obtain the following.

Theorem

Let a and b be non-zero integers with $\gcd(a,b) = d$. The equation

$$ax + by = d$$

has a solution (x_1, y_1) in integers which can be found by means of the extended Euclidean algorithm. Every solution of this equation can be obtained by substituting integers k into the formula

$$\left(x_1 + k\frac{b}{d}, y_1 - k\frac{a}{d}\right).$$